Decoherence and quantum mechanics in a gravitational field

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Abstract

Quantum mechanical interference of wave functions leads to some difficulties if a probability density is considered as a source of gravity. We show that an introduction of a quantum energy-momentum tensor as a source term in Einstein equations can be consistent with general relativity if the gravitational waves are quantized.

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1 Introduction

The conceptual difficulties with an introduction of quantum effects into Einstein gravity are well-known. It would be a pessimistic attitude to think that the inconsistencies can be avoided only in a complete quantum theory of gravity and matter . It has been shown that some semiclassical methods to combine classical gravity and quantum mechanics contradict experiments and intuition [1][2]. Page and Geilker [1] suggested that decoherence could help to solve these difficulties. An interaction with the environment as a source of the decoherence has been discussed in [5]-[6]. In ref.[3] Einstein gravity has been discussed and a phenomenological stochastic coupling has been introduced in order to achieve the decoherence.

Our aim in this paper is to discuss quantum mechanics combined with the linearized quantum gravity as a consistent approximation to a hypothetical complete quantum theory. First, let us outline a scenario implying a physical relevance of the quantized gravity. We consider a model of a gravitational field interacting with the quantum complex scalar field. If these fields result from a quantization of the classical field theory then they should satisfy the operator equations of motion ($\kappa = 8\pi G$, where G is the Newton constant)

$$\Box_g \psi = M^2 c^2 \psi \tag{1}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=-\kappa/c^4\Big(\partial_\mu\overline{\psi}\partial_\nu\psi+\partial_\nu\overline{\psi}\partial_\mu\psi$$
$$+\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\partial_\alpha\overline{\psi}\partial_\beta\psi+\partial_\beta\overline{\psi}\partial_\alpha\psi)-\frac{1}{2}g_{\mu\nu}M^2c^2\psi\overline{\psi}\Big) \tag{2}$$

where $g_{\mu\nu}$ is the Riemannian metric, $R_{\mu\nu}$ is the Ricci tensor constructed from the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ and the wave operator $\Box_g$ is

$$\Box_g=g^{\mu\nu}(\partial_\mu+g^{\alpha\beta}\Gamma_{\mu\alpha\beta})\partial_\nu \tag{3}$$

Subsequently we can consider a reduction of the quantum field theory of the scalar field to quantum mechanics . For this purpose we restrict the field equations to a fixed N-particle sector. Let ψ_{cl} be a solution of the Klein-Gordon equation. By means of the asymptotic fields we can define the one particle state $|\psi_{cl}\rangle$. Then, for the matrix elements $\langle 0|\psi(x)|\psi_{cl}\rangle$ we should obtain from eq.(1) approximately the Klein-Gordon equation in the metric g . Taking the expectation value of both sides of eq.(2) in the state $|\psi_{cl}\rangle$ we obtain the (quantum) Einstein equations determined by the energy–momentum tensor $\langle\psi_{cl}|T_{\mu\nu}(\psi)|\psi_{cl}\rangle$ of the quantum field theory.

At this stage it is also useful to see the non-relativistic limit of eqs.(1)-(2). The non-relativistic limit is simple if $g^{k0}=0$ and g^{00} is a slowly varying function of time. Then, we write

$$g^{00}=-1+V/c^2 \tag{4}$$

and

$$\psi=\exp(-\frac{iM}{\hbar}x_0)\tilde{\psi}$$

As a result we obtain an equation for $\tilde{\psi}$ (where $\triangle^{(3)}_{\mathcal{M}}$ is the threedimensional Laplacian on a Riemannian manifold $\mathcal{M}$)

$$i\hbar\partial_t\tilde{\psi}=(-\frac{\hbar^2}{2M}\triangle^{(3)}_{\mathcal{M}}-V)\tilde{\psi} \tag{5}$$

i.e., the conventional Schrödinger equation in the gravitational field. Then, in the limit $c\rightarrow\infty$ we obtain an equation for V

$$(\triangle^{(3)}_{\mathcal{M}}-\frac{1}{c^2}\partial_t^2)V=\kappa|\tilde{\psi}|^2$$

We consider a semiclassical approximation to the solution of the Klein-Gordon equation on a manifold. The semiclassical approximation is determined by the Hamiltonian

$$\mathcal{H}=\frac{1}{2}g^{\mu\nu}(q)p_\mu p_\nu \tag{6}$$

A solution of the Hamiltonian equations of motion determines a solution of the Klein-Gordon equation in the leading order of $\hbar$. The Hamilton-Jacobi function

$$W(x)=\int_x^y p_\mu dq^\mu$$

is computed on the trajectory starting from x and ending in y. We have in the semiclassical approximation

$$\psi(x) = \exp(\frac{i}{\hbar}\int_x^y p_\mu dq^\mu)\Big(\det(\frac{\partial q_\tau}{\partial x})\Big)^{\frac{1}{2}}\phi(q_\tau(x)) \qquad (7)$$

where q_τ is the trajectory starting from x at $\tau = 0$ and ending in y at τ. We have to exchange τ in favor of x_0 in eq.(7) .

The Hamiltonian equations have a constant of motion which we express in the form

$$\mathcal{H} = \frac{P^2}{2M}$$

For an approximate solution of the Hamiltonian equations it will be useful to introduce the frames e

$$e_a^\mu \eta^{ab} e_b^\nu = g^{\mu\nu} \qquad (8)$$

where $\eta = (1, 1, 1, -1)$ is the Minkowski metric. We express e by α

$$e_{\mu a} = \eta_{\mu c}(\exp\alpha)_a^c$$

Then, in the lowest order

$$e_{\mu a} = \eta_{\mu a} + \eta_{\mu c}\alpha_a^c \qquad (9)$$

Geodesic equations (which are equivalent to Hamiltonian equations) can be expressed in the form

$$d\Big(e_{\mu a}\frac{dq^\mu}{d\tau}\Big) = 0 \qquad (10)$$

$$de_{\mu a} + \Gamma^\nu_{\mu\alpha} e_{\nu a} dq^\alpha = 0 \qquad (11)$$

We solve these equations by iteration . So, we have from eq.(10)

$$\frac{dq^\mu}{d\tau} = -e^{\mu a} P_a/M \qquad (12)$$

In the zeroth order in α the trajectory is a straight line $q = x - P\tau/M$. We obtain a relation between τ and x_0 which can be applied in order to eliminate τ from W in eq.(7)

$$x_0 - y_0 = \tau P_0/M$$

2 Superposition principle and gravitational fluctuations

We suggest that a quantization of gravitational waves may help to reconcile classical Einstein gravity with quantum mechanics. The common way to introduce a quantum matter into the classical gravity [2] is a replacement of $T_{\mu\nu}$ by $\langle\psi|T_{\mu\nu}|\psi\rangle$. However, we encounter the Schrödinger cat problem with such an extension of classical gravity. We obtain a superposition of states with a certain density determined by $|\psi|^2$. In the classical gravity the metric is determined by the energy distribution (possibly random) , but not by the probability density of an interference of states of different energies [1].

We consider a perturbation expansion of eqs.(1)-(2) $g = g^{(cl)} + g^{(q)}$ with the approximation of a weak classical field g^{cl}. Then, $\Box g^{(q)} = 0$. Hence, the quantum gravitons are approximately moving as waves in a flat background. Then, expanding in $g^{(cl)}$

$$\Box_{g^{(q)}} g_{00}^{(cl)} - 2R_0^{(q)\nu} g_{0\nu}^{(cl)} + 2R_{0\mu\nu 0}^{(q)} g^{(cl)\mu\nu} = \frac{\kappa}{c^2}|\psi_1(t) + \psi_2(t)|^2 \tag{13}$$

where we take a superposition $\psi_1 + \psi_2$ of solutions of eq.(1). The observed value of the gravitational field is an average over the quantum fluctuations of the metric

$$\langle \Box_{g^{(q)}} g_{00}^{(cl)} - 2R_0^{(q)\nu} g_{0\nu}^{(cl)} + 2R_{0\mu\nu 0}^{(q)} g^{(cl)\mu\nu} \rangle = \frac{\kappa}{c^2}\langle|\psi_1(t) + \psi_2(t)|^2\rangle \tag{14}$$

In the linear approximation the l.h.s. of eq.(14) is $\Box g^{(cl)}$. In the semiclassical approximation for the solution of the Schrödinger equation we obtain a superposition of solutions of eq.(7)

$$\psi_1 + \psi_2 \to \exp(\frac{i}{\hbar}W^{(1)}(t))\phi(q^{(1)}(t)) + \exp(\frac{i}{\hbar}W^{(2)}(t))\phi(q^{(2)}(t))$$

where $q^{(k)}(t)$ are the classical trajectories. At the point x the probability density $\langle|\psi_t(x)|^2\rangle$ is equal to the diagonal part of the density matrix

$$\rho_t = \langle|\psi_t\rangle\langle\psi_t|\rangle \tag{15}$$

We assume that the quantum gravity consists of asymptotic fields α_a^b describing gravitational radiation and eventually some other degrees of freedom which do not have asymptotic fields. We quantize at the moment only the gravitational waves. In the linear approximation the metric as well as the tetrad e (α in the linear approximation) satisfy the wave equation

$$\Box\alpha_a^c = 0$$

We quantize the gravitational waves expanding α in the momentum space

$$\begin{aligned}&\alpha_a^b(x) = \sqrt{4\pi c}\sqrt{\hbar\kappa/c^4}(2\pi)^{-\frac{3}{2}} \int d\mathbf{k}|\mathbf{k}|^{-\frac{1}{2}} \exp(i\mathbf{kx})\\ &\sum_{\zeta=1,2} \Big(\mathcal{E}_a^b(\zeta,k)C(\zeta,\mathbf{k})\exp(-i|\mathbf{k}|x_0) + \overline{\mathcal{E}_a^b}(\zeta,k)C(\zeta,\mathbf{k})^+ \exp(i|\mathbf{k}|x_0)\Big)\end{aligned} \tag{16}$$

where

$$[C(\zeta,\mathbf{k}), C(\zeta',\mathbf{k}')^+] = \delta_{\zeta\zeta'}\delta(\mathbf{k}-\mathbf{k}')$$

are the creation and annihilation operators.

The Hamiltonian is

$$H_R = \int d\mathbf{k} \sum_{\zeta} c|\mathbf{k}| C(\zeta, \mathbf{k})^+ C(\zeta, \mathbf{k})$$

If the gravitational radiation is in equilibrium with light and matter then it should be described by the Gibbs distribution

$$\hat{\rho}_\beta = Z^{-1} \exp(-\beta H_R)$$

where $\frac{1}{\beta} = KT$, K is the Boltzmann constant and T denotes the temperature. In cosmological models the Gibbs distribution is believed to be correct when applied to the primordial gravitons (present at the earliest stages of the big bang) which had time to reach an equilibrium with other particles [7]. We suggest that these relict gravitons now reach the Earth and impose the classical behavior of large quantum systems. During an expansion of the Universe some gravitons are continuously being created as a result of a time-dependent gravitational field. These gravitons will not reach any equilibrium with the primordial ones. They have another energy distribution. The probability of graviton production is large at small wave number $\mathbf{k}$. Hence, the Planck distribution may be modified for small wave numbers $\mathbf{k}$ [8]. For these reasons we consider a more general density matrix $\hat{\rho}(H_R)$ as a function of the graviton energy H_R. We introduce a parameter $1/b$ (with the dimension of the inverse of the energy) as an energy cutoff . We could represent $\hat{\rho}$ by a Fourier-Laplace transform of the Gibbs distribution $\hat{\rho}_\mu$ (μ may be complex)

$$\hat{\rho} = \int d\mu \gamma(\mu) \hat{\rho}_\mu \tag{17}$$

Then, for a computation of expectation values we can use the methods applied for the Gibbs state , e.g., by a direct computation through an expansion in the number states we find in the Gibbs state

$$f_\beta^{PL}(\hbar c|\mathbf{k}|) \equiv \langle C^+(\mathbf{k}) C(\mathbf{k}) \rangle_\beta = \Big(\exp(\beta c \hbar |\mathbf{k}|) - 1 \Big)^{-1} \tag{18}$$

We can see that effectively $1/\beta$ plays the role of the energy cutoff in the Planck distribution because $f_\beta^{PL}(\hbar kc)$ decays exponentially fast when $kc\hbar > 1/\beta$. We shall often identify b with β in our discussion.

The correlation functions can be computed using eqs.(17) and (18)

$$\begin{gathered} Tr\Big(\alpha_c^a(t, \mathbf{x}) \alpha_{c'}^{a'}(0, \mathbf{x}') \hat{\rho} \Big) \equiv G_\beta(\mathbf{x}, \mathbf{x}'; t)_{cc'}^{aa'} = \\ \tfrac{\hbar\kappa}{2\pi^2 c^3} \int d\mathbf{k} \delta_{cc'}^{aa'} \tfrac{1}{|\mathbf{k}|} cos(\mathbf{k}(\mathbf{x} - \mathbf{x}')) \\ \Big((\tfrac{1}{2} + f_b(\hbar c|\mathbf{k}|)) cos(c|\mathbf{k}|t) - \tfrac{i}{2} sin(c|\mathbf{k}|t) \Big) \end{gathered} \tag{19}$$

where we denoted (a representation of the tensor δ and its properties are discussed in [4])

$$\delta^{ac}_{bd}(k) = \sum_{\zeta} \overline{\mathcal{E}^a_b}(k,\zeta)\mathcal{E}^c_d(k,\zeta)$$

In eq.(19) f_b is the graviton distribution. Our results for small time and space separations do not depend essentially on the form of f_b if

$$f_b(k) = \tilde{f}(bk)$$

and if $\tilde{f}$ decays sufficiently fast, e.g., $|\tilde{f}(k)| \leq Ak^{-6}$ for a large k. For a large time and large space separations the results depend on the singularity of $\tilde{f}(k)$ at $k = 0$ (for the Planck distribution $\tilde{f} \approx k^{-1}$). The distributions derived in inflationary models [12][13] behave powerlike in some intervals ,e.g. ,

$$\tilde{f}(k) = 0$$

if $k \geq 1$ and

$$\tilde{f}(k) = k^{-\sigma} \tag{20}$$

if $0 \leq k \leq 1$. Such a distribution leads to similar conclusions as the Planck distribution. However, an introduction of the infrared cutoff $\tilde{f}(k) = 0$ if $k \geq \epsilon$ would destroy the decoherence at a sufficiently large time.

An expectation value in the ground state χ of the free gravitational field is a special case of eq.(19) corresponding to the limit $\beta \to \infty$ (under the assumption $f_\infty(k) = 0$)

$$\begin{array}{l} G_\infty(\mathbf{x}, t; \mathbf{x}', 0) \equiv < \chi|\alpha(t,\mathbf{x})\alpha(0,\mathbf{x}')|\chi >= \\ \kappa c^{-4}\frac{\hbar c}{4\pi^2}\int d\mathbf{k}\frac{1}{|\mathbf{k}|}\cos(\mathbf{k}(\mathbf{x}-\mathbf{x}'))\exp(-ic|\mathbf{k}|t) \end{array} \tag{21}$$

Comparing eqs.(19) and (21) it can be seen that the first term on the r.h.s. of eq.(19) describes the zero point density (vacuum fluctuations) whereas the second one comes from the thermal gravitons in equilibrium with the environment. In general, the vacuum fluctuations cannot be neglected. After a renormalization they contribute to measurable effects. However, it can be shown [11] that renormalized vacuum fluctuations give a negligible contribution to the decoherence. They are small in comparison to the black body radiation at moderate temperatures. Moreover, the renormalized vacuum fluctuation part decreases to zero when the time becomes large. We subtract the vacuum fluctuations in eq.(19). After this subtraction the correlation function becomes real. We can associate a real random field α with such a correlation function (the field will be Gaussian in a linear approximation to gravity)

$$\langle \alpha^a_b(x)\alpha^c_d(x')\rangle = G^{ac}_{bd}(x-x')$$

where

$$G^{ac}_{bd}(x-x') = \hbar\kappa c^{-3}(2\pi)^{-2}\int d\mathbf{k}|\mathbf{k}|^{-1}\delta^{ac}_{bd}(k)\cos(\mathbf{k}(\mathbf{x}-\mathbf{x}'))\cos((x_0-x_0')|\mathbf{k}|)f_b(c|\mathbf{k}|\hbar) \tag{22}$$

In spherical coordinates $d\mathbf{k} = 2\pi d\theta \sin\theta dk k^2$. Hence, G_{th} can be expressed in the form (we skip the tensor δ)

$$G_{th} = 2\hbar\kappa c^{-3}|\mathbf{x}-\mathbf{x}'|^{-1}\pi^{-1}\int_0^\infty dk sin(k|\mathbf{x}-\mathbf{x}'|)\cos(ckt)f_b(\hbar ck)$$

3 Decoherent effect of gravitons

It is natural to associate gravitons with the decoherence. Gravitons interact with all particles. Hence, their decoherence effect would be universal. We consider the Einstein gravity for weak fields . We define the partial density matrix (averaged over the gravitons)

$$\rho_t(\mathbf{x},\mathbf{x}') = Tr_R\Big(\langle\mathbf{x}|\hat{\rho}(H_R)\exp(-i\frac{t}{\hbar}H_R)|\psi_t\rangle\langle\psi_t|\exp(i\frac{t}{\hbar}H_R)|\mathbf{x}'\rangle\Big) \tag{23}$$

The trace in eq.(23) can be obtained as an expectation value over the Gaussian random field α (or calculated in the operator formalism by means of the time-ordered products in the Fock space)

$$\langle\exp i\alpha J\rangle = \exp(-\frac{1}{2}JGJ)$$

where the Green functions G depend on the state under consideration. In particular, in the thermal state with subtracted vacuum fluctuations $G \rightarrow G_{th}$. If the vacuum fluctuations were to be taken into account then we would need to make the replacement $G_{th} \rightarrow G_\beta = G_{th} + G_\infty$ and subsequently $G_\infty \rightarrow G_F = i\triangle_F$ (in the notation of Bjorken and Drell[10]). However, the part $\int G_F d\mathbf{y}d\mathbf{y}$ contains infinities when the paths intersect . After a renormalization the remaining expression gives a negligible contribution to the decoherence [11].

We consider an initial state

$$\psi(\mathbf{x}) = \exp(i\mathbf{P}^{(1)}\mathbf{x}/\hbar)\phi(\mathbf{x}) + \exp(i\mathbf{P}^{(2)}\mathbf{x}/\hbar)\phi(\mathbf{x}) \tag{24}$$

We solve the geodesic equation (12) with the initial position x and the initial momenta $P^{(k)}$ by iteration. Then, till the first order in α

$$y_s^{(k)} = x - \frac{s}{M}P^{(k)} - \frac{1}{M}\int_0^s \alpha(x - \frac{\tau}{M}P^{(k)})P^{(k)}d\tau$$

The Green's function on the trajectory (till the zeroth order in α) is

$$\langle\alpha\left(y\left(\tau,x\right)\right)\alpha\left(y\left(s,x'\right)\right)\rangle = G(\mathbf{x}-\mathbf{x}' - \frac{\tau-s}{M}\mathbf{P}, x_0 - x_0' - \frac{\tau-s}{M}P_0)$$

In our approximation the time evolution is

$$\psi(\mathbf{x}) \rightarrow \psi_t(\mathbf{x}) = \exp(iP^{(1)}y_t^{(1)}/\hbar)\phi(\mathbf{y}_t^{(1)}) + \exp(iP^{(2)}y_t^{(2)}/\hbar)\phi(\mathbf{y}_t^{(2)})$$

where $y_t^{(k)}(x) = (\mathbf{y}_t^{(k)}(\mathbf{x}), ct)$ and

$$P^{(k)} = \Big(\mathbf{P}^{(k)}, \sqrt{M^2c^2 + (\mathbf{P}^{(k)})^2}\Big)$$

Then, a calculation of the trace (23) leads to the formula (we assume that ϕ is a slowly varying function, hence in its argument it is sufficient to calculate $y^{(k)}$ in the lowest order in α)

$$\begin{array}{l}
\langle| \psi_t(\mathbf{x}) |^2\rangle = |\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(1)})|^2 + |\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(2)})|^2 + \\
\Big(\overline{\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(2)})}\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(1)}) \exp\Big(\frac{i}{\hbar}(P^{(2)} - P^{(1)})x\Big) \\
+\overline{\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(1)})}\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(2)}) \exp\Big(- \frac{i}{\hbar}(P^{(2)} - P^{(1)}))x\Big) \\
\exp\Big(\frac{1}{M^2\hbar^2}\int_0^t \mathbf{P}^{(1)}\mathbf{P}^{(1)}G_{th}(\frac{s}{M}\mathbf{P}^{(1)} - \frac{\tau}{M}\mathbf{P}^{(2)}, P_0^{(1)}s/M - P_0^{(2)}\tau/M)\mathbf{P}^{(2)}\mathbf{P}^{(2)}dsd\tau \\
-\frac{1}{2M^2\hbar^2}\int_0^t \mathbf{P}^{(1)}\mathbf{P}^{(1)}G_{th}(\frac{s}{M}\mathbf{P}^{(1)} - \frac{\tau}{M}\mathbf{P}^{(1)}, P_0^{(1)}(s-\tau)/M)\mathbf{P}^{(1)}\mathbf{P}^{(1)}dsd\tau \\
-\frac{1}{2M^2\hbar^2}\int_0^t \mathbf{P}^{(2)}\mathbf{P}^{(2)}G_{th}(\frac{s}{M}\mathbf{P}^{(2)} - \frac{\tau}{M}\mathbf{P}^{(2)}, P_0^{(2)}(s-\tau)/M)\mathbf{P}^{(2)}\mathbf{P}^{(2)}d\tau ds\Big) \\
\equiv \rho_t^{(1)} + \rho_t^{(2)} + \rho_t^{(12)}
\end{array} \tag{25}$$

The notation $\mathbf{PP}$ is only symbolic, it means that we must sum the indices of $\mathbf{P}$ with the indices of the Green's function G_{th}.

If the off-diagonal terms vanish then the density of a superposition of wave functions is a sum of the densities. Classically the packets move apart and each of them is independently a source of a gravitational field. Without the decoherence there would be a superposition of the states of the packets.

It is easy to estimate the behavior of eq.(25) for a small time and either $\mathbf{P}^{(1)} \parallel \mathbf{P}^{(2)}$ or $\mathbf{P}^{(1)} \perp \mathbf{P}^{(2)}$. Let us denote in eq.(25)

$$|\rho^{(12)}| = 2\exp(-S_{12}(P))|\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(1)})\phi(\mathbf{x} - \frac{t}{M}\mathbf{P}^{(2)})|$$

where

$$S(P) = P^a P_b S^{bc}_{ad} P_c P^d$$

Then, for a small time we can set $s = \tau = 0$. There remains (under the assumption $f_b(u) = \tilde{f}(bu)$)

$$\begin{array}{l}
S_{12}(\mathbf{P}) = t^2M^{-2}\hbar^{-2}|(\mathbf{P}^{(1)})^2 - (\mathbf{P}^{(2)})^2|^2G_{th}(0,0) \\
= At^2M^{-2}\hbar^{-3}|(\mathbf{P}^{(1)})^2 - (\mathbf{P}^{(2)})^2|^2\kappa c^{-5}\beta^{-2}\int_0^\infty duuf_b(u/b) \\
= \tilde{A}|(\mathbf{P}^{(1)})^2 - (\mathbf{P}^{(2)})^2|(\frac{t}{M})^2 l_{dB}^{-2}|(\mathbf{P}^{(1)})^2 - (\mathbf{P}^{(2)})^2|\hbar^{-2}L_{PL}^2
\end{array} \tag{26}$$

In eq.(26) A and $\tilde{A}$ are constants of order 1, $l_{dB} = \hbar cb = l_C bmc^2$ will be called de Broglie length because if $\beta = b$ then $l_{dB} = \hbar c\beta$ is the wave length of a particle in a medium at temperature T, $l_C = \frac{\hbar}{mc}$ is the Compton length and m is the electron mass as our basic mass unit. Then, $\frac{\hbar}{|\mathbf{P}|}$ is particle's wave length at the momentum $\mathbf{P}$, $L_{PL} = \sqrt{\hbar\kappa/c^3}$ is the Planck length.

We can see that the decoherence is determined by the relation of the particle's wave length and the length of particle's trajectory to the Planck's length and de Broglie's length. For a large time the calculations are more involved. Let us denote by S_{12} the expression $S_{12}(P)$ without the fourlinear momenta. Then, we obtain a rather complicated formula for S_{12} (we omit the tensors δ)

$$\begin{aligned}
&S_{12} = \frac{\kappa}{2M^2c^4\hbar^2}\frac{\hbar c}{2\pi^2}\int d\mathbf{k}k^{-1}\tilde{f}\,(b\hbar ck) \\
&\Big(\left(\mathbf{P}^{(2)}\right)^2\left(\mathbf{k}\mathbf{P}^{(2)}/M+ck\right)^{-2}\left(1-\cos\left(t\mathbf{k}\mathbf{P}^{(2)}/M+tck\right)\right) \\
&+\left(\mathbf{P}^{(2)}\right)^2\left(\mathbf{k}\mathbf{P}^{(2)}/M-ck\right)^{-2}\left(1-\cos\left(t\mathbf{k}\mathbf{P}^{(2)}/M-tck\right)\right) \\
&+\left(\mathbf{P}^{(1)}\right)^2\left(\mathbf{k}\mathbf{P}^{(1)}/M+ck\right)^{-2}\left(1-\cos\left(t\mathbf{k}\mathbf{P}^{(1)}/M+tck\right)\right) \\
&+\left(\mathbf{P}^{(1)}\right)^2\left(\mathbf{k}\mathbf{P}^{(1)}/M-ck\right)^{-2}\left(1-\cos\left(t\mathbf{k}\mathbf{P}^{(1)}/M-tck\right)\right) \\
&+2(\mathbf{P}^{(1)}\mathbf{P}^{(2)})(\mathbf{P}^{(1)}\mathbf{k})(\mathbf{P}^{(2)}\mathbf{k})/M^2\left(c^2k^2-\left(\mathbf{P}^{(1)}\mathbf{k}\right)^2/M^2\right)^{-1}\left(c^2k^2-\left(\mathbf{P}^{(2)}\mathbf{k}\right)^2/M^2\right)^{-1} \\
&\left(1-\cos\left(t\mathbf{k}\left(\mathbf{P}^{(1)}-\mathbf{P}^{(2)}\right)/M\right)\right) \\
&-(\mathbf{P}^{(1)})^2(\mathbf{P}^{(2)})^2(ck+\mathbf{P}^{(1)}\mathbf{k}/M)^{-1}(ck+\mathbf{P}^{(2)}\mathbf{k}/M)^{-1}(1-\cos(ckt+t\mathbf{P}^{(1)}\mathbf{k}/M)) \\
&-(\mathbf{P}^{(1)})^2(\mathbf{P}^{(2)})^2(ck-\mathbf{P}^{(1)}\mathbf{k}/M)^{-1}(ck-\mathbf{P}^{(2)}\mathbf{k}/M)^{-1}(1-\cos(ckt-t\mathbf{P}^{(1)}\mathbf{k}/M)) \\
&-(\mathbf{P}^{(1)})^2(\mathbf{P}^{(2)})^2(ck+\mathbf{P}^{(1)}\mathbf{k}/M)^{-1}(ck+\mathbf{P}^{(2)}\mathbf{k}/m)^{-1}(1-\cos(ckt+t\mathbf{P}^{(2)}\mathbf{k}/M)) \\
&-(\mathbf{P}^{(1)})^2(\mathbf{P}^{(2)})^2(ck-\mathbf{P}^{(1)}\mathbf{k}/M)^{-1}(ck-\mathbf{P}^{(2)}\mathbf{k}/M)^{-1}(1-\cos(ckt-t\mathbf{P}^{(2)}\mathbf{k}/M))\Big)
\end{aligned} \tag{27}$$

It can be seen from eq.(27) that if $\tilde{f}(0)=0$ then owing to the Lebesgue theorem $S_{12}(t)$ is bounded in t. If $\tilde{f}(k)$ is singular at $k=0$ then $S_{12}(t)$ grows for a large t. In order to investigate this case in more detail let us consider first the Planck distribution f_β^{PL} . We neglect $\mathbf{P}/Mc$ in the denominator of eq.(27) and consider either $\mathbf{P}^{(1)}\parallel\mathbf{P}^{(2)}$ or $\mathbf{P}^{(1)}\perp\mathbf{P}^{(2)}$. Then, we obtain

$$\begin{aligned}
&S_{12}(P) = \frac{\kappa}{M^2c^4\hbar^2}\frac{\hbar}{c\pi}|(\mathbf{P}^{(2)})^2-(\mathbf{P}^{(1)})^2|^2\int_0^\infty\frac{dk}{k}(\exp(\beta\hbar ck)-1)^{-1}\left(1-\cos\left(tck\right)\right) \\
&= \frac{\kappa}{M^2c^4\hbar^2}\frac{\hbar}{c\pi}|(\mathbf{P}^{(2)})^2-(\mathbf{P}^{(1)})^2|^2ct\int_0^1 d\alpha\int_0^\infty dk(\exp(\beta\hbar ck)-1)^{-1}\sin(\alpha ckt) \\
&= \frac{\kappa}{M^2c^4\hbar^2}\frac{\hbar t}{\pi}|(\mathbf{P}^{(2)})^2-(\mathbf{P}^{(1)})^2|^2\int_0^1 d\alpha\left(\frac{\pi}{2\beta\hbar c}\coth\left(\frac{\pi\alpha t}{\beta\hbar}\right)-\frac{1}{2\alpha c}\right) \\
&= \frac{\kappa}{M^2c^4\hbar^2}\frac{\hbar}{2c\pi}|(\mathbf{P}^{(2)})^2-(\mathbf{P}^{(1)})^2|^2\ln\left(\frac{\beta\hbar}{\pi t}\sinh\left(\frac{t\pi}{\beta\hbar}\right)\right) \\
&\approx \frac{ct}{2l_{dB}}L_{PL}^2\hbar^{-2}|(\mathbf{P}^{(2)})^2-(\mathbf{P}^{(1)})^2|^2M^{-2}c^{-2}
\end{aligned} \tag{28}$$

for a large t. Here, the formula 3.911 of Gradshtein and Ryzhik [9] has been applied

$$\int_0^\infty du\sin(au)\Big(\exp(\beta u)-1\Big)^{-1}=\frac{\pi}{2\beta}\coth\left(\frac{\pi a}{\beta}\right)-\frac{1}{2a}$$

For the inflationary distribution (20) $\frac{ct}{l_{dB}}$ in eq.(28) is replaced by $\left(\frac{ct}{l_{dB}}\right)^\sigma$.

As a result of the decoherence the gravitational field is determined by the density distribution of two separate quantum particles. Hence, we have obtained a classical addition of probabilities instead of the quantum addition of amplitudes. This conclusion follows from the results (25)-(28) which should be inserted into eq.(14).

Finally, let us discuss numerical estimates on some expressions in this paper. First, let us assume that the graviton distribution is determined by the Planck distribution (as suggested by Weinberg [7]). The decoherence time depends on the ratio of the Planck length to the de Broglie wave length (as could have been expected because we have two universal length scales at finite temperatures). To see the meaning of l_{dB} let us write $KT = mv_T^2$ then $l_{dB} = l_C\frac{c^2}{v_T^2}$ (the Compton length multiplied by the ratio of the rest energy to the kinetic energy). For numerical estimates it is useful to introduce the Planck temperature T_{PL} determined by $L_{PL} = l_{dB}$, then $\beta^2 c^5 = \kappa/\hbar$ and $L_{PL}/l_{dB} = T/T_{PL}$ (note that $T_{PL} = 1.3 \times 10^{32} Kelvin$). Instead of the de Broglie length we could use the temperature independent length scale : the Compton length l_C (choosing the electron mass m as a mass unit). The Compton length comes in a natural way because the ratio of the gravitational force to the Coulomb force between electrons is

$$F_{grav}/F_{Coul} = \frac{1}{\frac{e^2}{\hbar c}}(L_{PL}/l_C)^2 \approx \frac{1}{\frac{e^2}{\hbar c}}10^{-46}$$

In particular, we can write for a small time

$$\rho_t \approx \exp\Big(-\frac{M^2}{m^2}(\frac{L_{PL}}{l_C})^2 l_{dB}^{-2}(ct)^2(\frac{P}{Mc})^4\Big)$$

The exponential is of the order $(10^{-23}\frac{M}{m})^2(\frac{v}{c})^4$. We can see that we need the number of particles $N \equiv \frac{M}{m} \approx 10^{23}$ if the decoherence is to be visible .

If instead of the Planck distribution we have the (simplified) inflationary distribution (20) then the behavior of S_{12} is determined by the integral (with a certain constant B)

$$S_{12} = B\int_0^b dk k^{-\sigma-1}\left(1-\cos\left(kct\right)\right)$$

Under the assumption $0 \leq \sigma < 2$ this integral behaves as $(\frac{ct}{l_{dB}})^\sigma$ for a large t and as $(\frac{ct}{l_{dB}})^2$ for a small t (we use the notation $l_{dB} = bc\hbar$). In inflationary models σ is different in various frequency ranges and for the lowest frequency range $\sigma = 2$. For such a σ the integral S_{12} is infrared divergent. At the large wave length $1/k$ there can be a sharp cutoff or a continuous change of the behavior of f leading to the finite integral for S_{12}. This is a sensitive problem in the spectral theory of gravitational waves [14][15] concerning waves with length larger than the actual Hubble horizon length. It is interesting that in this way the decoherence is associated with the prospective evolution of the Universe.

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